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The general solution and Faxén laws for the temperature fields in and outside an isolated ellipsoid

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Abstract. The image systems for the disturbance temperature fields in and outside an isolated ellipsoid driven by an n-th order ambient field are introduced and their connection to the ellipsoidal harmonics is derived. More general ambient fields may be handled by superposition of these basic solutions. These result have been used to derive the Faxén relation for the arbitrary n-th order multipole moment. The explicit expressions for the temperature fields and the thermal moment tensors for an ellipsoid in linear and quadratic ambient fields are given to illustrate the method.

1. Introduction

The prediction of the effective transport properties of a two-phase system has been a classic and important problem [1–3]. For example, the determination of the effective thermal conductivity of a composite material consisting of spherical inclusions dispersed in a continuous matrix is well established [4–7]. We present here a solution form suitable for ellipsoidal inclusions with the goal of providing the microscopic description of the pre-averaged system. The ellipsoidal geometry is chosen because it embraces a wide class of nonspherical inclusions, from slender bodies in fiber-enhanced plastic to flat disks in clay suspensions.

To predict the effective transport property accurately to $O(c^2)$, where c is the volume fraction occupied by the inclusion, the interaction between two particles should be determined [1, 7, 8]. For nonspherical inclusions, the pair-interaction problem cannot be solved analytically and a numerical approach is necessary, augmented where available by asymptotic solutions. When the test pair are at large separations, the so called method-of-reflection provides an accurate solution [9–12]. At smaller separations, the method converges too slowly and an another approach is required. Among these, the boundary collocation method has gained in popularity in recent years [13–15]. The results of this paper provide the necessary information for both the method-of-reflections solution and the boundary collocation technique.

The essence of the boundary collocation technique is summarized as follows. One chooses a set of basis functions that satisfy the governing equation identically. The desired solution is written as truncated expansion in these basis functions with the coefficients determined by matching the boundary conditions at chosen points (the collocation points). The end result is a set of linear equations, the solution of which provides the coefficients and the expansion solution. *The temperature fields derived in the present work form a suitable set of basis functions for this technique.*

The method-of-reflection was developed by Smoluchowski [16]. It is essentially an iterative solution of the pair-interaction problem in which the disturbance field produced by one particle is used to correct the ambient field at the other particle (and so forth). In the usual terminology, the ambient field of interest is called the *incident field* and the resulting disturbance field is called the *reflected field*. Thus, given an incident field, we need a method for calculating the reflected field. One approach, which is readily applied to non-spherical particles, expands the reflected field in a multipole expansion with the moments determined from the Faxén laws [17].

In the recent literature, a variety of Faxén laws has been derived for different physical problems (velocity problem, temperature problem, etc.), different particle shapes (sphere, spheroid, ellipsoid), and various multipole moments [7, 8, 18–23]. One intriguing aspect of the Faxén relation which is not widely appreciated is the functional link between the Faxén relations and the singularity solutions for so called *conjugate* problems. For example, the Faxén law for the thermal dipole and the disturbance field forced by a linear ambient field possess the same functional form. This duality was first noted by Hinch [24] and a derivation is given in Kim [25] for rigid particles (or perfect conductors). The proof is more difficult for the general two-phase problem but has been derived recently by Kim and Lu [26].

The paper is divided as follows. In Section 2, we derive expressions for the temperature field driven by an n-th order ambient field. In Section 3, we use the duality proof from [26] to derive the Faxén relations for the n-th order thermal multipole moment. Finally, the explicit solutions for problems with linear and quadratic ambient fields are presented in Section 4 to illustrate the ideas presented in Sections 2 and 3.

2. Solution for arbitrary ambient field

We consider a source-free temperature field in and outside an isolated ellipsoid in an ambient temperature field $T^{\infty}(\mathbf{x})$. We assume that $T^{\infty}(\mathbf{x})$ satisfies the Laplace equation. The coordinate system is chosen so that the equation for the surface of the ellipsoid is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a \geqslant b \geqslant c,$$

where a, b, c are the three semi-axes of the ellipsoid.

The interior and exterior temperature fields satisfy the Laplace equation and the boundary conditions are continuity of the temperature and the normal heat flux across the particle surface. In addition, the exterior temperature field approaches the ambient temperature field when $\mathbf{x}$ is far away from the particle. They may be described in mathematical form as follows.

Governing equations:

$$\nabla^2 T_{out}(\mathbf{x}) = 0, \quad \mathbf{x} \in \text{matrix}; \tag{2.1a}$$

$$\nabla^2 T_{in}(\mathbf{x}) = 0, \quad \mathbf{x} \in \text{particle}. \tag{2.1b}$$

Boundary conditions:

$$T_{out}(\mathbf{x}_s) = T_{in}(\mathbf{x}_s), \tag{2.2a}$$

$$k_1 \mathbf{n} \cdot \nabla T_{out}(\mathbf{x}_s) = k_2 \mathbf{n} \cdot \nabla T_{in}(\mathbf{x}_s), \tag{2.2b}$$

$$T_{out}(\mathbf{x}) = T^{\infty}(\mathbf{x}) \text{ as } |\mathbf{x}| \to \infty, \tag{2.2c}$$

where k_1 and k_2 are the thermal conductivities of the matrix and the particle respectively, $\mathbf{n}$ is the outward unit vector normal to the particle surface, and $\mathbf{x}_s$ is a point on the particle surface.

Our goal is to show that for the n-th order ambient field, $W_{k_1 k_2 \ldots k_n} x_{k_1} x_{k_2} \ldots x_{k_n}$, the exterior solution may be written as a simple distribution of singularities,

$$T_{out}(\mathbf{x}) = T^{\infty}(\mathbf{x}) + \sum_{m=0}^{[(n-1)/2]} L_{(n-2m)} \int_{\mathbf{E}} \frac{f_{(n-2m+1)}(\mathbf{x}')}{4\pi k_1 |\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}\mathbf{A}(\mathbf{x}').$$

The details, including the region of distribution $\mathbf{E}$, the density function $f_{(n)}$ and the type of singularities employed are discussed below.

We start with ellipsoidal coordinates and the ellipsoidal harmonics. The ellipsoidal coordinates (ϱ, μ, ν) are the solutions of the cubic equation

$$\frac{x^2}{\lambda} + \frac{y^2}{\lambda - h^2} + \frac{z^2}{\lambda - k^2} = 1, \tag{2.3}$$

for fixed values of (x, y, z), where

$$k^2 = a^2 - c^2, \; h^2 = a^2 - b^2.$$

The three roots of (2.3) are chosen so that

$$\infty > \varrho^2 \geqslant k^2, \; k^2 \geqslant \mu^2 \geqslant h^2, \; h^2 \geqslant \nu^2 \geqslant 0.$$

The three surfaces, $\varrho =$ constant, $\mu =$ constant and $\nu =$ constant, consisting of confocal ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets respectively, form a triply-orthogonal coordinate system. The transformation between the Cartesian and the ellipsoidal coordinates is given by

$$x^2 = \frac{\varrho^2 \mu^2 \nu^2}{h^2 k^2},$$

$$y^2 = \frac{(\varrho^2 - h^2)(\mu^2 - h^2)(h^2 - \nu^2)}{h^2(k^2 - h^2)},$$

$$z^2 = \frac{(\varrho^2 - k^2)(k^2 - \mu^2)(k^2 - \nu^2)}{k^2(k^2 - h^2)}.$$

The internal ellipsoidal harmonic $E_n^m(\varrho)E_n^m(\mu)E_n^m(v)$ is regular at the origin and the external ellipsoidal harmonic $F_n^m(\varrho)E_n^m(\mu)E_n^m(v)$ is regular at infinity [27]. Here, E_n^m denotes the Lamé function of the first kind and F_n^m is the Lamé function of the second kind, of degree n and order m, where n and m are nonnegative and positive integers respectively with $n \leqslant 2n + 1$. The internal ellipsoidal harmonic is a polynomial in $\mathbf{x}$, thus we can rewrite a polynomial in $\mathbf{x}$ of degree n in terms of linear combinations of internal ellipsoidal harmonics.

Our approach for an arbitrary ambient field problem is based on the expansion of the arbitrary ambient field in a Taylor series about a reference point (usually the center of the particle). The governing equations and boundary conditions are linear so that the solution for a linear combination of ambient fields is the sum of the solutions for each sub-problem. Thus we need only solve the sub-problem involving the n-th order ambient field. Such a field can always be expressed in terms of the internal ellipsoidal harmonics. The exterior solution then can be written in terms of the external harmonics with the same "angular dependence", i.e.,

$$T_{out}(\mathbf{x}) = \sum_{l=1}^{n}\sum_{m=1}^{2l+1} A_l^m E_l^m(\varrho)E_l^m(\mu)E_l^m(v) + \sum_{l=1}^{n}\sum_{m=1}^{2l+1} B_l^m F_l^m(\varrho)E_l^m(\mu)E_l^m(v),$$

$$T_{in}(\mathbf{x}) = \sum_{l=1}^{n}\sum_{m=1}^{2l+1} A_l^m E_l^m(\varrho)E_l^m(\mu)E_l^m(v) + \sum_{l=1}^{n}\sum_{m=1}^{2l+1} C_l^m E_l^m(\varrho)E_l^m(\mu)E_l^m(v),$$

where the first terms on the RHS are just the ambient field, while the second terms are the exterior and interior disturbance fields generated by the presence of the ellipsoid. The A_l^m's may be determined by rewriting the given ambient temperature field into a linear combination of the internal ellipsoidal harmonics. Thus there remain only two sets of unknown coefficients, B_l^m's and C_l^m's, which may be determined through the application of the two boundary conditions, (2.2a) and (2.2b). The result is:

$$B_l^m = \left. \frac{A_l^m E_l^m(\varrho)(\mathrm{d}E_l^m(\varrho)/\mathrm{d}\varrho)(k_2 - k_1)}{k_1 E_l^m(\varrho)(\mathrm{d}F_l^m(\varrho)/\mathrm{d}\varrho) - k_2 F_l^m(\varrho)(\mathrm{d}E_l^m(\varrho)/\mathrm{d}\varrho)} \right|_{\varrho=a}, \tag{2.4a}$$

$$C_l^m = \left. \frac{A_l^m F_l^m(\varrho)(\mathrm{d}E_l^m(\varrho)/\mathrm{d}\varrho)(k_2 - k_1)}{k_1 E_l^m(\varrho)(\mathrm{d}F_l^m(\varrho)/\mathrm{d}\varrho) - k_2 F_l^m(\varrho)(\mathrm{d}E_l^m(\varrho)/\mathrm{d}\varrho)} \right|_{\varrho=a}, \tag{2.4b}$$

We note that, for an n-th order field (say $W_{k_1k_2\ldots k_n}x_{k_1}\ldots x_{k_n}$), the solution involves only ellipsoidal harmonics of degree $n, n-2, \ldots$ down to 1(0) if n is odd (even).

According to the theorems by Miloh [28], there are two kinds of integral representations for exterior ellipsoidal harmonic depending on whether the harmonic is even or odd in z. It can be shown that the external ellipsoidal harmonic, when even in z, contains terms like

$$\int_{\mathrm{E}} \frac{(x')^c(y')^d q^{-1}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}(\mathbf{x}') \quad \text{with } c + d = n, n-2, \ldots 1(0), \text{ if } n \text{ is odd (even)}. \tag{2.5}$$

When odd in z, the ellipsoidal harmonic contains the following terms:

$$\frac{\partial}{\partial z}\int_{\mathbf{E}} \frac{(x')^e(y')^f q(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}(\mathbf{x}') \quad \text{with } e+f = n-1, n-3, \ldots 0(1), \text{ if } n \text{ is odd (even).} \tag{2.6}$$

The $e + f$ in the above equation is even (odd) if n is odd (even), and the function $q(x', y')$ is defined as

$$q(x', y') = \left[1 - \frac{x'^2}{a_E^2} - \frac{y'^2}{b_E^2}\right]^{1/2}$$

with

$$a_E = (a^2 - c^2)^{1/2}, \; b_E = (b^2 - c^2)^{1/2}.$$

$\mathbf{E}(x', y')$, the integration domain, is the interior of the fundamental ellipse, which is the degenerate elliptical disk in a family of confocal ellipsoids, defined as follows:

$$\frac{x'^2}{a_E^2} + \frac{y'^2}{b_E^2} = 1, \quad z' = 0.$$

It should be mentioned that q^{-1} is the requisite charge distribution over the fundamental ellipse which generates ellipsoidal equipotential surfaces. From here on, the functional dependency of q and dA on $\mathbf{x}'$ will be dropped unless otherwise noted.

We now derive an alternate form for terms in (2.5) and (2.6) without the x' and y' appearing in the integrands. Without loss of generality, we examine the following term:

$$\int_{\mathbf{E}} \frac{(x')^s q^t}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}. \tag{2.7}$$

As shown in Appendix 1, repeated integration by parts of the above expression generates terms of the form

$$\int_{\mathbf{E}} \frac{q^{s+t}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}, \; \frac{\partial^2}{\partial x^2}\int_{\mathbf{E}} \frac{q^{s+t+2}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}, \ldots, \frac{\partial^s}{\partial x^s}\int_{\mathbf{E}} \frac{q^{2s+t}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA} \quad \text{if } s \text{ is even,}$$

$$\frac{\partial^1}{\partial x^1}\int_{\mathbf{E}} \frac{q^{s+t+1}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}, \; \frac{\partial^3}{\partial x^3}\int_{\mathbf{E}} \frac{q^{s+t+3}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}, \ldots, \frac{\partial^s}{\partial x^s}\int_{\mathbf{E}} \frac{q^{2s+t}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA} \quad \text{if } s \text{ is odd.}$$

The more general case (2.5) follows in an analogous manner but with mixed derivatives. The terms containing derivatives with respect to z may be generated from (2.6) and from the following relation

$$\frac{\partial^{n-2}}{\partial x_{k_1} \ldots \partial x_{k_{n-2}}} \frac{\partial^2}{\partial z^2}\int_{\mathbf{E}} \frac{q^n}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA} = -\frac{\partial^{n-2}}{\partial x_{k_1} \ldots \partial x_{k_{n-2}}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\int_{\mathbf{E}} \frac{q^n}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA},$$

where the fact that $\int_{\mathbf{E}}(q^n/|\mathbf{x} - \mathbf{x}'|)\mathrm{dA}$ is harmonic (see Appendix 2) has been used. Thus for an n-th order ambient field, the solution contains contributions from external ellipsoidal harmonics of degree $n, n - 2 \ldots$ and, for each harmonic, there are terms like those in (2.8) with $t = -1$ and $s = n, n - 2, \ldots$ respectively.

We digress here to introduce a transformation relation established in [29] between the singularity integral and the external ellipsoidal harmonic. Define

$$H_n(\mathbf{x}) \equiv \frac{1}{2n + 1} \int_{\mathbf{E}} \frac{f_{(n+1)}}{|\mathbf{x} - \mathbf{x}'|} \mathrm{dA} \tag{2.9a}$$

and

$$G_n(\mathbf{x}) \equiv \int_{\lambda}^{\infty} \left(\frac{x^2}{(a^2 + t)} + \frac{y^2}{(b^2 + t)} + \frac{z^2}{(c^2 + t)} - 1 \right)^n \frac{\mathrm{d}t}{\Delta(t)}, \tag{2.9b}$$

where

$$f_{(n)} = \frac{2n - 1}{2\pi a_E b_E} q^{2n-3},$$

$$\Delta(t) = [(a^2 + t)(b^2 + t)(c^2 + t)]^{1/2}.$$

The relation is

$$H_n(\mathbf{x}) = \frac{(-1)^n (2n)!}{2^{2n+1} n! n!} G_n(\mathbf{x}). \tag{2.9c}$$

A proof by mathematical induction is given in [29]. (The constant of proportionality in (2.9c) may be established by considering the spherical case). With this relation, the expression for the exterior field can be transformed readily from terms involving singularity integrals to terms involving external ellipsoidal harmonics.

For the n-th order ambient field, adding all contributions together, we obtain the following representation for the exterior temperature field:

$$T_{out} = T^{\infty} + \frac{1}{4\pi k_1} \sum_{m=0}^{[(n-1)/2]} \sum_{k=1}^{[(n+1-2m)/2]} \times \frac{(-1)^{n-k+1}(2n - 4m - 2k + 3)!}{2^{2n-4m-2k+3}[(n - 2m - k + 1)!]^2} L^{(k)}_{(n-2m-2k+2)} G_{n-2m-k+1},$$

or in terms of singularity integrals,

$$T_{out} = T^{\infty} + \sum_{m=0}^{[(n-1)/2]} \sum_{k=1}^{[(n+1-2m)/2]} L^{(k)}_{(n-2m-2k+2)} \int_{\mathbf{E}} \frac{f_{(n-2m-k+2)}}{4\pi k_1 |\mathbf{x} - \mathbf{x}'|} \mathrm{dA}, \tag{2.10b}$$

where

$$L^{(k)}_{(n)} = \frac{(-1)^n}{n!} P^{(k)}_{k_1 k_2 \ldots k_n} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_n}}. \tag{2.10c}$$

n is odd

$n = 1$		∂G_1	
$n = 3$	∂G_2	$\partial^3 G_3$	
$n = 5$	∂G_3	$\partial^3 G_4$	$\partial^5 G_5$
	. . . .		
	. . . . .		

n is even

$n = 2$		$\partial^2 G_2$	
$n = 4$	$\partial^2 G_3$	$\partial^4 G_4$	
$n = 6$	$\partial^2 G_4$	$\partial^4 G_5$	$\partial^6 G_6$
	. . . .		
	. . . . .		

where $$\partial^l \equiv \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_l}$$

Fig. 1. The solution components in equation (2.10a). For a certain n, the solution contains terms above the corresponding line, e.g. ∂G_1, ∂G_2, and $\partial^3 G_3$ are the solution components when $n = 3$.

We note here that the monopoles are excluded because we are interested in source-free problems. The $P^{(k)}$'s are the multipole moments, e.g., $P_i^{(k)}$ and $P_{ij}^{(k)}$ are the thermal dipole and quadrupole respectively. In Fig. 1, we show the components of the exterior solution in terms of G_n functions diagrammatically.

We derive an alternate form for the exterior solution which involves only a single summation. First of all, we introduce the identity (see Appendix 1)

$$\int_E \frac{q^n}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A} = \frac{n}{n+2}\int_E \frac{q^{n-2}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A} - \frac{1}{(n+2)^2}\left(a_E^2\frac{\partial^2}{\partial x^2} + b_E^2\frac{\partial^2}{\partial y^2}\right)\int_E \frac{q^{n+2}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A},$$

where n is a positive integer. With this identity, one can eventually convert all terms into the terms appearing in the far-right diagonal of Fig. 1, e.g., $\partial^1 G_2$ can be converted into the $\partial^1 G_1$ and $\partial^3 G_3$ terms. Thus, only a single summation describes the solution and one obtains the following solution form for the exterior field:

$$T_{out}(\mathbf{x}) = T^{\infty}(\mathbf{x}) + \frac{1}{4\pi k_1}\sum_{m=0}^{[(n-1)/2]} \frac{(-1)^n(2n-4m+1)!}{2^{2n-4m+1}[(n-2m)!]^2} L_{(n-2m)} G_{n-2m}(\mathbf{x}), \qquad (2.11a)$$

or in terms of singularity integrals,

$$T_{out}(\mathbf{x}) = T^{\infty}(\mathbf{x}) + \sum_{m=0}^{[(n-1)/2]} L_{(n-2m)} \int_{\mathbf{E}} \frac{f_{(n-2m+1)}}{4\pi k_1 |\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}\mathbf{A}, \tag{2.11b}$$

where

$$L_{(n)} = \frac{(-1)^n}{n!} P_{k_1 k_2 \ldots k_n} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_n}}, \tag{2.11c}$$

$$P_{k_1 k_2 \ldots k_l} = \sum_{k=1}^{[(n-l+2)/2]} P^{(k)}_{k_1 k_2 \ldots k_l}. \tag{2.11d}$$

We are now in a position to examine the behavior of the exterior solution on the particle surface and propose a solution form for the interior field. It has been shown in [29] that

$$\partial^n G_n = \sum_{m=0}^{[n/2]} 2^{n-m} \frac{n!}{m!} \left(\delta_{k_1 k_2} \cdots \delta_{k_{2m-1} k_{2m}} x_{k_{2m+1}} \cdots x_{k_n} \right.$$
$$\left. \times \int_{\lambda}^{\infty} \frac{F^m}{[\Pi_{\alpha=1}^{n} (\alpha_{k_\alpha}^2 + t)]} \frac{\mathrm{d}t}{\Delta(t)} + \ldots (\mathrm{sym}) \right), \quad \alpha \neq k_2, k_4, \ldots k_{2m},$$

where $F = x^2/(a^2 + t) + y^2/(b^2 + t) + z^2/(c^2 + t) - 1$, δ_{ij} is the Kronecker delta function, and the symbols a_1, a_2, a_3 are used to represent a, b, c. At the summation index m, there are $n!/(2^m m!(n - 2m)!)$ terms corresponding to all possible permutations of $\{k_1 \ldots k_n\}$ in the representative term. When evaluated on the particle surface $\lambda = 0$, the exterior solution with even(odd) n-th order ambient field is a polynomial in $\mathbf{x}$ of degree n with only even(odd) degree terms. Therefore, we propose the following solution form for the interior field:

$$T_{in}(\mathbf{x}) = T^{\infty}(\mathbf{x}) + \sum_{m=0}^{[(n-1)/2]} L^I_{(n-2m)} + T^c, \tag{2.12a}$$

where

$$L^I_{(n)} = \frac{(-1)^n}{n!} P^I_{k_1 \ldots k_n} x_{k_1} \cdots x_{k_n}, \tag{2.12b}$$

$$T^c = D^I_{k_1 \ldots k_n} W_{k_1 \ldots k_n} \quad \text{if } n \text{ is even}$$
$$= 0 \quad \text{if } n \text{ is odd}. \tag{2.12c}$$

In (2.12c), $D^I_{k_1 \ldots k_n}$ is a constant tensor depending on the shape parameters of the particle (a, b, c) and the thermal conductivities of the system, k_1 and k_2. For even n, the temperature at the center differs by T^c from that of the ambient field. The form for T^c follows from linearity. We note that, without the introduction of the G_n function, the interior solution is an n-th degree polynomial of $\mathbf{x}$ with only even(odd) degree terms for even(odd) n-th order ambient field.

We close this section by one remark concerning the constraints for the multipole moments. As shown in Appendix 2,

$$C^{(k)}_{iik_3\ldots k_r} P^{(k)}_{iik_3\ldots k_r} = 0 \quad \text{for all } k \text{ and } r.$$

In the above equation, $C^{(k)}_{k_1k_2\ldots k_r}$ is the number associated with each $P^{(k)}_{k_1k_2\ldots k_r}$ to describe how many identical $P^{(k)}_{k_1k_2\ldots k_r}$ exist due to the permutation symmetry, e.g. $C^k_{12} = 2$ and $C^k_{123} = 6$, etc.. The result may be justified in the following way. Without loss of generality, we can set $P^{(k)}_{iik_3\ldots k_n}$ zero by redefining the lower order tensor $P^{(k-1)}_{k_3\ldots k_n}$. Similarly, we may set $P^{(k-1)}_{iik_5\ldots k_n}$ to be zero, by redefining the lower order tensor $P^{(k-2)}_{k_7\ldots k_n}$, and so forth. In contrast, the multipole moments in (2.11a, b), in general, will not satisfy these constraints. Thus in certain computations, equation (2.10b) provides a more useful starting point.

3. Faxén relations

We are now in a position to derive the Faxén relation for the n-th order moment, using the recent developments presented in [26]. We cite some important concepts and results of that paper for our purposes here. The interested reader may consult that paper for further details.

The n-th order moment of the thermal flux is defined as

$$\begin{aligned}
\tilde{P}_{k_1\ldots k_n} &\equiv \frac{k_2 - k_1}{k_2} \int_{\mathbf{S}_p} \mathbf{F}(\mathbf{x}_s) \cdot \mathbf{n} x_{S_{k_1}} \ldots x_{S_{k_n}} \, \mathrm{d}\mathbf{S}(\mathbf{x}_S) \\
&= (k_1 - k_2) \int_{\mathbf{S}_p} \nabla T(\mathbf{x}_S) \cdot \mathbf{n} x_{S_{k_1}} \ldots x_{S_{k_n}} \, \mathrm{d}\mathbf{S}(\mathbf{x}_S) \\
&= n(k_1 - k_2) \int_{\mathbf{V}_p} x_{k_1} \ldots x_{k_{n-1}} \frac{\partial T(\mathbf{x})}{\partial x_{k_n}} \, \mathrm{d}\mathbf{V}(\mathbf{x})
\end{aligned}$$

with $\mathbf{S}_p$ and $\mathbf{V}_p$ representing the surface and the volume of the particle respectively. Here, $\mathbf{F}$ is the heat flux defined as $-k_2 \nabla T$. The divergence theorem has been used in the above derivation to change the integral domain from particle surface to particle volume. The moments of the thermal flux appear in the multipole expansion and the lower-order moments appear directly as the physical quantities of interest in the calculations of the effective transport properties.

We consider a reference temperature field $T'(\mathbf{x})$ with the associated ambient field $T^{\infty\prime}(\mathbf{x})$ and the temperature field of interest $T(\mathbf{x})$ with associated ambient field $T^{\infty}(\mathbf{x})$. We start with a modified form of Green's second identity [26],

$$(k_1 - k_2) \int_{\mathbf{V}_p} \nabla T(\mathbf{x}) \cdot \nabla T^{\infty\prime}(\mathbf{x}) \, \mathrm{d}V(\mathbf{x}) = -k_1 \int_{\mathbf{V}_p^+} \nabla^2 T'(\mathbf{x}) \, T^{\infty}(\mathbf{x}) \, \mathrm{d}\mathbf{V}(\mathbf{x}). \tag{3.1}$$

The notation $\mathbf{V}_p^+$ indicates explicitly that the integrand is evaluated using the exterior solution. We now take $T^{\infty\prime}$ as an n-th order ambient field $W_{k_1\ldots k_n} x_{k_1} \ldots x_{k_n}$, then

$$\begin{aligned}
\text{LHS of equation (3.1)} &= n(k_1 - k_2) W_{k_1\ldots k_n} \int_{\mathbf{V}_p} x_{k_1} \ldots x_{k_{n-1}} \frac{\partial T(\mathbf{x})}{\partial x_{k_n}} \, \mathrm{d}\mathbf{V}(\mathbf{x}) \\
&= W_{k_1\ldots k_n} \tilde{P}_{k_1\ldots k_n}.
\end{aligned}$$

From the previous section, we know that $-k_1\nabla^2 T'(\mathbf{x})$ is identically

$$\sum_{m=0}^{[(n-1)/2]} \frac{(-1)^{n-2m}}{(n-2m)!} P_{k_1\ldots k_{n-2m}} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_{n-2m}}} \int_{\mathrm{E}} f_{(n-2m+1)} \delta(\mathbf{x}-\mathbf{x}')\, \mathrm{d}\mathbf{A}.$$

After changing the order of integration, the RHS of (3.1) becomes

$$\sum_{m=0}^{[(n-1)/2]} \frac{(-1)^{n-2m}}{(n-2m)!} P_{k_1\ldots k_{n-2m}} \int_{\mathrm{E}} f_{(n-2m+1)}(\mathbf{x}')$$

$$\left[\int_{\mathrm{V}_p^+} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_{n-2m}}} \delta(\mathbf{x}-\mathbf{x}') T^{\infty}(\mathbf{x})\, \mathrm{d}\mathbf{V}(\mathbf{x})\right] \mathrm{d}\mathbf{A}(\mathbf{x}').$$

We introduce the material tensors $\mathbf{Z}$ which fix the linear relations between the $\mathbf{P}$'s and $\mathbf{W}$ as follows,

$$P_{k_1\ldots k_{n-2m}} = Z_{(k_1\ldots k_{n-2m})(l_1\ldots l_n)} W_{l_1\ldots l_n}, \quad m = 0, 1, 2 \ldots [(n-1/2],$$

and an identity from the theory of distributions,

$$\int_{\mathrm{V}_p^+} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_{n-2m}}} \delta(\mathbf{x}-\mathbf{x}') T^{\infty}(\mathbf{x})\, \mathrm{d}\mathbf{V}(\mathbf{x}) = (-1)^{n-2m} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_{n-2m}}} T^{\infty}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}'}.$$

We then obtain an expression for $\tilde{P}_{l_1\ldots l_n}$ as

$$\sum_{m=0}^{[(n-1/2]} \frac{1}{(n-2m)!} Z_{(k_1\ldots k_{n-2m})(l_1\ldots l_n)} \int_{\mathrm{E}} f_{(n-2m+1)} \frac{\partial}{\partial x_{k_1}} \cdots \frac{\partial}{\partial x_{k_{n-2m}}} T^{\infty}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}'}\, \mathrm{d}\mathbf{A}. \tag{3.2}$$

Thus the n-th order moment is given exactly by an integration, over the focal ellipse, of the gradients of the ambient field.

The above relation may also be written in the symbolic operator form of Brenner and Haber [23], which is particularly useful in situations where the ambient field is expressed analytically. An identity derived by Kim and Arunachalam [29] (their equation (26)) applies here as well, so that

$$\begin{aligned}
\int_{\mathrm{E}} f_{(n)} (\nabla)^{n-1} T^{\infty}\, \mathrm{d}\mathbf{A} &= \frac{(2n)!}{n!} \sum_{k=1}^{\infty} \frac{(k+n-1)!D^{2k-2}}{(k-1)!(2k+2n-2)!} (\nabla)^{n-1} T^{\infty}(\mathbf{x}')|_{\mathbf{x}'=\mathbf{x}_c} \\
&= \frac{(2n)!}{n!} \sum_{k=1}^{\infty} \frac{(k+n-1)!\tilde{D}^{2k-2}}{(k-1)!(2k+2n-2)!} (\nabla)^{n-1} T^{\infty}(\mathbf{x}')|_{\mathbf{x}'=\mathbf{x}_c} \\
&= \frac{(2n)!}{2^n n!} \left\{\left(\frac{1}{\tilde{D}} \frac{\partial}{\partial \tilde{D}}\right)^{n-1} \left(\frac{\sinh \tilde{D}}{\tilde{D}}\right)\right\} (\nabla)^{n-1} T^{\infty}(\mathbf{x}')|_{\mathbf{x}'=\mathbf{x}_c},
\end{aligned}$$

where

$$D^2 = a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2} + c^2 \frac{\partial^2}{\partial z^2}$$

$$= \tilde{D}^2 + c^2 \nabla^2$$

with

$$\tilde{D}^2 = a_E^2 \frac{\partial^2}{\partial x^2} + b_E^2 \frac{\partial^2}{\partial y^2}.$$

In the last step of the above derivation, the fact that the temperature field satisfies the Laplace equation has been used to reduce D^2 to $\tilde{D}^2$. We thus obtain an expression for $\tilde{P}_{l_1 \ldots l_n}$ in terms of symbolic operators as

$$\sum_{m=0}^{[(n-1)/2]} \frac{(2n - 4m + 2)!}{(n - 2m)!\,(n - 2m + 1)!\,2^{n-2m+1}} Z_{(k_1 \ldots k_{m-2m})(l_1 \ldots l_n)}$$

$$\times \left\{ \left(\frac{1}{\tilde{D}} \frac{\partial}{\partial \tilde{D}} \right)^{n-2m} \left(\frac{\sinh \tilde{D}}{\tilde{D}} \right) \right\} (\nabla)^{n-2m} T^{\infty}(\mathbf{x}')|_{\mathbf{x}' = \mathbf{x}_c}. \tag{3.3}$$

4. Solutions for linear and quadratic fields

In this section, we derive the explicit solutions for the linear and quadratic ambient fields, i.e. problems with $T^{\infty} = \mathbf{G} \cdot \mathbf{x}$ and $\mathbf{H}:\mathbf{xx}$ respectively. As mentioned in Section 2, we need only solve simpler sub-problems and construct the full solution by summing all sub-solutions. In addition, the cyclic symmetry between sub-solutions with respect to the dependency on $\{a, b, c\}$, $\{x, y, z\}$, and $\{1, 2, 3\}$ may be employed to achieve a further reduction of the problem. For example, the solution for $T^{\infty} = G_y y$ may be generated by cycling the solution for $T^{\infty} = G_x x$. The linear and quadratic ambient fields may be expanded as follows:

$$\mathbf{G} \cdot \mathbf{x} = G_x x + G_y y + G_z z, \tag{4.1a}$$

$$\mathbf{H}:\mathbf{xx} = \frac{H_{11}}{3}(2x^2 - y^2 - z^2) + \frac{H_{22}}{3}(-x^2 + 2y^2 - z^2) + \frac{H_{33}}{3}(-x^2 - y^2 + 2z^2)$$

$$+ (H_{12} + H_{21})xy + (H_{13} + H_{31})xz + (H_{23} + H_{32})yz. \tag{4.1b}$$

It should be noted that in the above expansion we group the terms involving x^2, y^2, z^2 in such a way that the grouping matches the functional form of the corresponding internal ellipsoidal harmonic of degree 2 and furthermore we can utilize the cyclic symmetry of the solutions for the first three ambient fields in the RHS of (4.1b). Thus we need only solve problems with $T^{\infty} = x$, $2x^2 - y^2 - z^2$, and xy respectively.

(1) $T^\infty(\mathbf{x}) = x$:

If we choose $E_1^1(\mu) = \mu$, then

$$T^\infty = x = \frac{E_1^1(\varrho)E_1^1(\mu)E_1^1(\nu)}{hk}.$$

The A_1^1 is thus determined and we obtain the following solution immediately,

$$T_{out}(\mathbf{x}) = A_1^1 E_1^1(\varrho)E_1^1(\mu)E_1^1(\nu) + B_1^1 F_1^1(\varrho)E_1^1(\mu)E_1^1(\nu),$$

$$T_{in}(\mathbf{x}) = A_1^1 E_1^1(\varrho)E_1^1(\mu)E_1^1(\nu) + C_1^1 E_1^1(\varrho)E_1^1(\mu)E_1^1(\nu),$$

where B_1^1 and C_1^1 are constants which can be determined by (2.4a, b)(Note that T_{out} approaches T^∞ because of the decay in the external ellipsoidal harmonics).

After some straightforward manipulations and the aid of the theorems by Miloh [28], we obtain the following solution,

$$T_{out}(\mathbf{x}) = T^\infty(\mathbf{x}) + \frac{8\pi k_1(k_2 - k_1)}{3\left(\int_0^\infty \frac{\mathrm{d}t}{(a^2+t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}\right)} \frac{\partial}{\partial x}\int_{\mathbf{E}} \frac{f_{(2)}}{4\pi k_1|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A},$$

$$T_{in}(\mathbf{x}) = T^\infty(\mathbf{x}) + \frac{-(k_2 - k_1)\int_0^\infty \frac{\mathrm{d}t}{(a^2+t)\Delta(t)}}{\int_0^\infty \frac{\mathrm{d}t}{(a^2+t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}}\, x.$$

As mentioned previously, the solutions for $T^\infty = G_y y$ and $G_z z$ may be obtained by the mnemonic of cycling the suffices and the dependence on a, b, and c. We may construct the full solution by summing the solutions of the sub-problems. The final solution is

$$T_{out}(\mathbf{x}) = T^\infty(\mathbf{x}) - \mathbf{S}\cdot\nabla\int_{\mathbf{E}} f_{(2)}(\mathbf{x}')\frac{1}{4\pi k_1|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}\,(\mathbf{x}'), \tag{4.2a}$$

$$T_{in}(\mathbf{x}) = T^\infty(\mathbf{x}) - \mathbf{S}^I\cdot\mathbf{x}, \tag{4.2b}$$

where S is the strength of the dipole moment of the thermal disturbance produced by the ellipsoid. We use a similar symbol $\mathbf{S}^I$ in interior solution to reveal the similarity of the solution form between interior and exterior solutions. The solution form is consistent with the general results of Section 2.

The $\mathbf{S}$ and $\mathbf{S}^I$ may be rewritten as $\mathbf{M}\cdot\mathbf{G}$ and $\mathbf{M}^I\cdot\mathbf{G}$ respectively because of the linearity of $\mathbf{S}$ and $\mathbf{S}^I$ to $\mathbf{G}$. The components of the diagonal second-order tensors $\mathbf{M}$ and $\mathbf{M}^I$ are listed in Appendix 3. The $\mathbf{M}$ and $\mathbf{M}^I$ are material tensors which depend only on the shape parameters of the particle and the thermal conductivities of the system. Substituting (4.2a) into (3.2), we obtain, after dropping $\mathbf{G}$ which is arbitrary, the Faxén law:

$$\mathbf{S} = \mathbf{M}\cdot\int_{\mathbf{E}} f_{(2)}(\mathbf{x}')\nabla T^\infty(\mathbf{x})|_{\mathbf{x}=\mathbf{x}'}\,\mathrm{d}\mathbf{A}\,(\mathbf{x}'),$$

or in terms of symbolic operators,

$$\mathbf{S} = 3\mathbf{M}\cdot\left\{\left(\frac{1}{\tilde{D}}\frac{\partial}{\partial\tilde{D}}\right)\frac{\sinh\tilde{D}}{\tilde{D}}\right\}\nabla T^{\infty}(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_c}.$$

(2) $T^{\infty} = xy$:

We first rewrite the ambient field in terms of the internal ellipsoidal harmonic as given by

$$xy = \frac{E_2^3(\varrho)E_2^3(\mu)E_2^3(\nu)}{hk^2\sqrt{k^2-h^2}}$$

with

$$E_2^3(\mu) = \mu\sqrt{k^2-\mu^2}.$$

The expressions for T_{out} and T_{in} involve two undetermined constants B_2^3 and C_2^3 as before. We also need the identity (Kim and Arunachalam [29], see Appendix 1)

$$\int_{\mathrm{E}} x_j' q^n \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA} = -\frac{(a_j^2-c^2)}{(n+2)}\frac{\partial}{\partial x_j}\int_{\mathrm{E}} q^{(n+2)}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA}, \quad (n \geqslant -1)$$

which may be established using integration by parts. We then obtain the following solution,

$$T_{out}(\mathbf{x}) = T^{\infty}(\mathbf{x}) + \frac{8\pi k_1(k_1-k_2)/15}{\displaystyle\int_0^{\infty}\frac{\mathrm{d}t}{(a^2+t)(b^2+t)\Delta(t)}(k_2-k_1) + \frac{2k_1}{(a^2+b^2)abc}}$$
$$\times\ \frac{\partial^2}{\partial x\partial y}\int_{\mathrm{E}} f_{(3)}\frac{1}{4\pi k_1|\mathbf{x}-\mathbf{x}'|}\,\mathrm{dA},$$

$$T_{in}(\mathbf{x}) = T^{\infty}(\mathbf{x}) + \frac{\displaystyle(k_1-k_2)\int_0^{\infty}\frac{\mathrm{d}t}{(a^2+t)(b^2+t)\Delta(t)}}{\displaystyle\int_0^{\infty}\frac{\mathrm{d}t}{(a^2+t)(b^2+t)\Delta(t)}(k_2-k_1) + \frac{2k_1}{(a^2+b^2)abc}}\,xy.$$

(3) $T^{\infty} = 2x^2 - y^2 - z^2$:

This case is much harder than the previous ones because of the coupling of two internal ellipsoidal harmonics. Rewriting the ambient field in terms of internal ellipsoidal harmonics and choosing

$$E_2^1(\mu) = \mu^2 + \tfrac{1}{6}(p_1-4)(h^2+k^2)$$

and

$$E_2^2(\mu) = \mu^2 + \tfrac{1}{2}(p_2-4)(h^2+k^2),$$

where p_1 and p_2 are the roots of the following equation:

$$(h^2 + k^2)p(p - 4) + 12h^2k^2 = 0,$$

we reach the following expression for $T^\infty(\mathbf{x})$:

$$T^\infty(\mathbf{x}) = A^{1x}_{2x}E^1_2(\varrho)E^1_2(\mu)E^1_2(\nu) + A^{2x}_{2x}E^2_2(\varrho)E^2_2(\mu)E^2_2(\nu) + C^x,$$

where A^{1x}_{2x}, A^{2x}_{2x} and C^x are functions of the particle geometry parameters and are listed in Appendix 1. For the present problem, we have four undetermined constants B^{1x}_{2x}, B^{2x}_{2x}, C^{1x}_{2x}, and C^{2x}_{2x}. We obtain four equations by collecting terms for the two ellipsoidal harmonics in the two boundary conditions.

By introducing a new identity, equation (A-2) in Appendix 1, we finally obtain the following solution for T_{out}:

$$T_{out}(\mathbf{x}) = T^\infty(\mathbf{x}) + \left(d_1\frac{\partial^2}{\partial x^2} + d_2\frac{\partial}{\partial y^2}\right)\int_{\mathbf{E}}\frac{q^3}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A} + d_3\int_{\mathbf{E}}\frac{(q^{-1} - 3q)}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A},$$

where d_1, d_2, and d_3 are functions of the shape parameters of the particle and the thermal conductivities of the system. We may eliminate the third term by using the identity

$$\int_{\mathbf{E}}\frac{(n(n - 2)q^{n-4} - n^2q^{n-2})}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A} = \left(a_E^2\frac{\partial^2}{\partial x^2} + b_E^2\frac{\partial}{\partial y^2}\right)\int_{\mathbf{E}}\frac{q^n}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A}$$

where n is positive integer. As mentioned in Section 2, we can generate the $\partial^2/\partial z^2$ term from the $\partial^2/\partial x^2$ and $\partial/\partial y^2$ terms to construct the full solution. Thus we attain our goal of expressing the final solution in the form:

$$T_{out}(\mathbf{x}) = T^\infty(\mathbf{x}) + \tfrac{1}{2}\mathbf{Q}:\nabla\nabla\int_{\mathbf{E}}\frac{f_{(3)}}{4\pi k_1|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}\mathbf{A},$$

where $\mathbf{Q}$ is the thermal quadrupole.

The derivation for the interior solution is straightforward but tedious. After summing all interior solutions of sub-problems, we obtain the following result:

$$T_{in}(\mathbf{x}) = T^\infty(\mathbf{x}) + \tfrac{1}{2}\mathbf{Q}^I:\mathbf{xx} + C^I:\mathbf{H}.$$

Again, we can rewrite $\mathbf{Q}$ and $\mathbf{Q}^I$ as $\mathbf{N}:\mathbf{H}$ and $\mathbf{N}^I:\mathbf{H}$, respectively, because of the linearity of $\mathbf{Q}$ and $\mathbf{Q}^I$ to the tensor $\mathbf{H}$. The $\mathbf{N}$ and $\mathbf{N}^I$ are also material tensors. The quadrupole $\mathbf{Q}$ may be deduced by the same procedure used for $\mathbf{S}$. The results are as follows:

$$\mathbf{Q} = \tfrac{1}{2}\mathbf{N}:\int_{\mathbf{E}} f_{(3)}\nabla\nabla T^\infty(\mathbf{x})|_{\mathbf{x}=\mathbf{x}'}\,\mathrm{d}\mathbf{A},$$

or in terms of symbolic operators,

$$\mathbf{Q} = \frac{15}{2}\mathbf{N}:\left\{\left(\frac{1}{\tilde{D}}\frac{\partial}{\partial\tilde{D}}\right)^2\frac{\sinh\tilde{D}}{\tilde{D}}\right)\nabla\nabla T^\infty(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_c}.$$

The explicit solution for $T^\infty = U_{ijk}x_i x_j x_k$ has also been derived by the authors using procedures analogous to those employed for linear and quadratic fields. Again, the results are consistent with the general pattern derived in Section 2. The interested reader may contact the authors for details.

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Appendix 1

The derivations for some useful identities are presented here. The derivatives of $q(\mathbf{x}')$ with respect to x_i' give

$$x_i' q^n = -\frac{(a_i^2 - c^2)}{(n+2)}\frac{\partial q^{n+2}}{\partial x_i'}. \tag{A.1}$$

We now deduce the following relation

$$\int_E \frac{x_i' x_j' q^n}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{dA} = -\frac{(a_i^2 - c^2)}{(n+2)}\frac{\partial}{\partial x_i}\int_E \frac{x_j' q^{n+2}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{dA}.$$

In the derivation of the above relation, two facts have been used:
(1)

$$q(x', y')|_{x' = \pm a_E(1 - y'^2/b_E^2)^{1/2}} = q(x', y')|_{y' = \pm b_E(1 - x'^2/a_E^2)^{1/2}} = 0$$

(2)

$$\frac{\partial}{\partial x_k'}\frac{1}{|\mathbf{x} - \mathbf{x}'|} = -\frac{\partial}{\partial x_k}\frac{1}{|\mathbf{x} - \mathbf{x}'|}.$$

Repeating the same procedure one more time, we obtain

$$\int_E \frac{x_i' x_j' q^n}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{dA} = \delta_{ij}\frac{(a_i^2 - c^2)}{(n+2)}\int_E \frac{q^{n+2}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{dA}$$

$$+ \frac{(a_i^2 - c^2)(a_j^2 - c^2)}{(n+2)(n+4)}\frac{\partial^2}{\partial x_i \partial x_j}\int_E \frac{q^{n+4}}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{dA}. \tag{A.2}$$

With the above identity, the following relations may be deduced:

$$\int_{\mathbf{E}} \frac{q^n}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} = \int_{\mathbf{E}} q^{n-2}\left(1-\frac{x'^2}{a_E^2}-\frac{y'^2}{b_E^2}\right)\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}$$

$$= \int_{\mathbf{E}} \frac{q^{n-2}-(2/n)q^n}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} - \frac{1}{n(n+2)}\left(a_E^2\frac{\partial^2}{\partial x^2}+b_E^2\frac{\partial^2}{\partial y^2}\right)\int_{\mathbf{E}} \frac{q^{n+2}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}.$$

Collecting like terms, we then reach the final result

$$\int_{\mathbf{E}} \frac{q^n}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} = \frac{n}{n+2}\int_{\mathbf{E}} \frac{q^{n-2}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} - \frac{1}{(n+2)^2}\left(a_E^2\frac{\partial^2}{\partial x^2}+b_E^2\frac{\partial^2}{\partial y^2}\right)\int_{\mathbf{E}} \frac{q^{n+2}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}.$$

or in terms of the G_n function,

$$G_n = \frac{n}{n+2}G_{n-1} - \frac{1}{(n+2)^2}\left(a_E^2\frac{\partial^2}{\partial x^2}+b_E^2\frac{\partial^2}{\partial y^2}\right)G_{n+1}.$$

We now examine what happens to $\int_{\mathbf{E}}(x'^n q^m/|\mathbf{x}-\mathbf{x}'|)\mathrm{d}\mathbf{A}$. Again, with (A.1) as a starting equation, we apply the method of integration by parts repeatedly to obtain:

$$\int_{\mathbf{E}} x'^n q^m \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}$$

$$= \frac{a_E^2}{(m+2)}\left((n-1)\int_{\mathbf{E}}\frac{x'^{n-2}q^{m+2}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} - \frac{\partial}{\partial x}\int_{\mathbf{E}}\frac{x'^{n-1}q^{m+2}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}\right)$$

$$= \frac{(a_E^2)^2}{(m+2)(m+4)}\left((n-1)(n-3)\int_{\mathbf{E}}\frac{x'^{n-4}q^{m+4}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} - (2n-3)\frac{\partial}{\partial x}\int_{\mathbf{E}}\frac{x'^{n-3}q^{m+4}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A} + \frac{\partial^2}{\partial x^2}\int_{\mathbf{E}}\frac{x'^{n-2}q^{m+4}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}\right)$$

$$\vdots$$

$$= \sum_{l=0}^{n/2} C_l \frac{\partial^l}{\partial x^l}\int_{\mathbf{E}}\frac{(x')^l q^{m+n}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}, \quad \text{if } n \text{ is even}$$

or

$$\sum_{l=1}^{(n+1)/2} C_l \frac{\partial^l}{\partial x^l}\int_{\mathbf{E}}\frac{(x')^{l-1} q^{m+n+1}}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}\mathbf{A}, \quad \text{if } n \text{ is odd,}$$

where the C_l's are constants and functions of n, m, and a_E. We obtain the results as listed in (2.8) by applying the method of integration by parts repeatedly to those terms which still involve x' in the integrand.

Appendix 2

(1) Taking the Laplacian of the singularity integral, we obtain

$$\nabla^2 \int_{\mathbf{E}} \frac{f_{(n)}}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{dA} = \int_{\mathbf{E}} f_{(n)} \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \mathrm{dA}$$

$$= 4\pi \int_{\mathbf{E}} f_{(n)} \delta(\mathbf{x} - \mathbf{x}') \, \mathrm{dA} = 0$$

In the last step, we have made use of the fact that **x**, which is a point outside or on the ellipsoid, never meets **x**′, which is a point on the fundamental ellipse.

(2) Consider the following disturbance field:

$$T^D(\mathbf{x}) = C^{(k)}_{iik_3 \ldots k_n} P^{(k)}_{iik_3 \ldots k_n} \frac{\partial^n}{\partial x_i^2 \partial x_{k_3} \ldots \partial x_{k_n}} \int_{\mathbf{E}} f_{(n+1)} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{dA}.$$

$$(\text{sum of } i,\ k_3 \ldots k_n \in \{1, 2, 3\}) \qquad \text{(A.3)}$$

We define a set of new coefficients $\mathbf{C}^{(k)'}\mathbf{P}^{(k)'}$ as given by

$$C^{(k)'}_{iik_3 \ldots k_n} P^{(k)'}_{iik_3 \ldots k_n} = C^{(k)}_{iik_3 \ldots k_n} P^{(k)}_{iik_3 \ldots k_n} - \tfrac{1}{3} C^{(k)}_{jjk_3 \ldots k_n} P^{(k)}_{jjk_3 \ldots k_n}.$$

(no sum on i, but sum on j)

Replacing the $\mathbf{C}^{(k)}\mathbf{P}^{(k)}$ with the $\mathbf{C}^{(k)'}\mathbf{P}^{(k)'}$ in (A.3), the disturbance field then becomes

$$C^{(k)}_{iik_3 \ldots k_n} P^{(k)}_{iik_3 \ldots k_n} \frac{\partial^n}{\partial x_i^2 \partial x_{k_3} \ldots \partial x_{k_n}} \int_{\mathbf{E}} f_{(n+1)} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{dA}$$

$$- \tfrac{1}{3} C^{(k)}_{iik_3 \ldots k_n} P^{(k)}_{jjk_3 \ldots k_n} \frac{\partial^{n-2}}{\partial x_{k_3} \ldots \partial x_{k_n}} \left(\frac{\partial^2}{\partial x_k^2} \right) \int_{\mathbf{E}} f_{(n+1)} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{dA}.$$

(sum of i, j, k)

We have proven that the Laplacian part of the above expression vanishes. Therefore, with the new coefficient set, the disturbance field does not change. We note that the summation of the new coefficient set, $C^{(k)}_{iik_3 \ldots k_n} P^{(k)'}_{iik_3 \ldots k_n}$ (sum on i), is just zero. We thus conclude that, without loss of generality, we can set the summation of $C^{(k)}_{iik_3 \ldots k_n} P^{(k)}_{iik_3 \ldots k_n}$ over i equal to zero by redefining the lower-order tensor. An important special case of this concept is that we may set the trace of the quadrupole, $Q_{ii} = 0$ (sum on i).

Appendix 3

We present here the full information on the non-zero components of $\mathbf{M}$, $\mathbf{M}^I$, $\mathbf{N}$, and $\mathbf{N}^I$.

$$M_{11} = \frac{8\pi k_1(k_2 - k_1)}{3\left(\int_0^\infty \frac{\mathrm{d}t}{(a^2 + t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}\right)},$$

$$M_{22} = \frac{8\pi k_1(k_2 - k_1)}{3\left(\int_0^\infty \frac{\mathrm{d}t}{(b^2 + t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}\right)},$$

$$M_{33} = \frac{8\pi k_1(k_2 - k_1)}{3\left(\int_0^\infty \frac{\mathrm{d}t}{(c^2 + t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}\right)},$$

$$M_{11}^I = \frac{(k_2 - k_1)\int_0^\infty \frac{\mathrm{d}t}{(a^2 + t)\Delta(t)}}{\int_0^\infty \frac{\mathrm{d}t}{(a^2 + t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}},$$

$$M_{22}^I = \frac{(k_2 - k_1)\int_0^\infty \frac{\mathrm{d}t}{(b^2 + t)\Delta(t)}}{\int_0^\infty \frac{\mathrm{d}t}{(b^2 + t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}},$$

$$M_{33}^I = \frac{(k_2 - k_1)\int_0^\infty \frac{\mathrm{d}t}{(c^2 + t)\Delta(t)}}{\int_0^\infty \frac{\mathrm{d}t}{(c^2 + t)\Delta(t)}(k_2 - k_1) + \frac{2k_1}{abc}},$$

$$N_{1111} = -\frac{C_c}{9}\left(\frac{A_{2x}^{1x}C_1}{\int_{s_1}} + \frac{A_{2x}^{2x}C_2}{\int_{s_2}}\right),$$

$$N_{2222} = -\frac{C_c}{9}\left(\frac{A_{2y}^{1y}C_3}{\int_{s_1}} + \frac{A_{2y}^{2y}C_4}{\int_{s_2}}\right),$$

$$N_{3333} = -\frac{C_c}{9}\left(\frac{A_{2z}^{1z}C_5}{\int_{s_1}} + \frac{A_{2z}^{2z}C_6}{\int_{s_2}}\right),$$

$$N_{1122} = -\frac{C_c}{9}\left(\frac{A_{2y}^{1y}C_1}{\int_{s_1}} + \frac{A_{2y}^{2y}C_2}{\int_{s_2}}\right) = N_{2211},$$

$$N_{1133} = -\frac{C_c}{9}\left(\frac{A_{2z}^{1z}C_1}{\int_{s_1}} + \frac{A_{2z}^{2z}C_2}{\int_{s_2}}\right) = N_{3311},$$

$$N_{2233} = -\frac{C_c}{9}\left(\frac{A_{2z}^{1z}C_3}{\int_{s_1}} + \frac{A_{2z}^{2z}C_4}{\int_{s_2}}\right) = N_{3322},$$

$$N_{1212} = -\frac{C_c}{2}\,\frac{1}{\displaystyle\int_\lambda^\infty \frac{(k_2 - k_1)\,\mathrm{d}t}{(a^2 + t)(b^2 + t)\Delta(t)} + \frac{2k_1}{(a^2 + b^2)abc}} = N_{2112} = N_{2121} = N_{1221},$$

$$N_{1313} = -\frac{C_c}{2}\,\frac{1}{\displaystyle\int_\lambda^\infty \frac{(k_2 - k_1)\,\mathrm{d}t}{(a^2 + t)(c^2 + t)\Delta(t)} + \frac{2k_1}{(a^2 + c^2)abc}} = N_{3113} = N_{3131} = N_{1331},$$

$$N_{2323} = -\frac{C_c}{2}\,\frac{1}{\displaystyle\int_\lambda^\infty \frac{(k_2 - k_1)\,\mathrm{d}t}{(b^2 + t)(c^2 + t)\Delta(t)} + \frac{2k_1}{(b^2 + c^2)abc}} = N_{3223} = N_{3232} = N_{2332},$$

$$N_{1111}^I = -C_c'\left(\frac{A_{2x}^{1x}B_1\int'_{s_1}}{\int_{s_1}} + \frac{A_{2x}^{2x}B_2\int'_{s_2}}{\int_{s_2}}\right),$$

$$N_{2222}^I = -C_c'\left(\frac{A_{2y}^{1y}A_1\int'_{s_1}}{\int_{s_1}} + \frac{A_{2y}^{2y}A_2\int'_{s_2}}{\int_{s_2}}\right),$$

$$N_{3333}^I = -C_c'\left(\frac{A_{2z}^{1z}(-A_1 - B_1)\int'_{s_1}}{\int_{s_1}} + \frac{A_{2z}^{2z}(-A_2 - B_2)\int'_{s_2}}{\int_{s_2}}\right),$$

$$N_{1122}^I = -C_c'\left(\frac{A_{2y}^{1y}B_1\int'_{s_1}}{\int_{s_1}} + \frac{A_{2y}^{2y}B_2\int'_{s_2}}{\int_{s_2}}\right) = N_{2211}^I,$$

$$N_{1133}^I = -C_c'\left(\frac{A_{2z}^{1z}B_1\int'_{s_1}}{\int_{s_1}} + \frac{A_{2z}^{2z}B_2\int'_{s_2}}{\int_{s_2}}\right) = N_{3311}^I,$$

$$N_{2233}^I = -C_c'\left(\frac{A_{2z}^{1z}A_1\int'_{s_1}}{\int_{s_1}} + \frac{A_{2z}^{2z}A_2\int'_{s_2}}{\int_{s_2}}\right) = N_{3322}^I,$$

$$N^I_{1212} = \frac{-(k_2 - k_1)\int_\lambda^\infty \dfrac{\mathrm{d}t}{(a^2+t)(b^2+t)\Delta(t)}}{\int_\lambda^\infty \dfrac{(k_2-k_1)\,\mathrm{d}t}{(a^2+t)(b^2+t)\Delta(t)} + \dfrac{2k_1}{(a^2+b^2)abc}} = N^I_{2112} = N^I_{2121} = N^I_{1221},$$

$$N^I_{1313} = \frac{-(k_2 - k_1)\int_\lambda^\infty \dfrac{\mathrm{d}t}{(a^2+t)(c^2+t)\Delta(t)}}{\int_\lambda^\infty \dfrac{(k_2-k_1)\,\mathrm{d}t}{(a^2+t)(c^2+t)\Delta(t)} + \dfrac{2k_1}{(a^2+c^2)abc}} = N^I_{3113} = N^I_{3131} = N^I_{1331},$$

$$N^I_{2323} = \frac{-(k_2 - k_1)\int_\lambda^\infty \dfrac{\mathrm{d}t}{(b^2+t)(c^2+t)\Delta(t)}}{\int_\lambda^\infty \dfrac{(k_2-k_1)\,\mathrm{d}t}{(b^2+t)(c^2+t)\Delta(t)} + \dfrac{2k_1}{(b^2+c^2)abc}} = N^I_{3223} = N^I_{3232} = N^I_{2332},$$

$$C^I_{11} = \frac{C'_c}{6}\left(\frac{A^{1x}_{2x}A_1\int'_{s_1}}{\int_{s_1}} + \frac{A^{2x}_{2x}A_2\int'_{s_2}}{\int_{s_2}}\right)(k^2-h^2) + \frac{C'_c}{6}\left(\frac{A^{1x}_{2x}B_1\int'_{s_1}}{\int_{s_1}} + \frac{A^{2x}_{2x}B_2\int'_{s_2}}{\int_{s_2}}\right)k^2,$$

$$C^I_{22} = \frac{C'_c}{6}\left(\frac{A^{1y}_{2y}A_1\int'_{s_1}}{\int_{s_1}} + \frac{A^{2y}_{2y}A_2\int'_{s_2}}{\int_{s_2}}\right)(k^2-h^2) + \frac{C'_c}{6}\left(\frac{A^{1y}_{2y}B_1\int'_{s_1}}{\int_{s_1}} + \frac{A^{2y}_{2y}B_2\int'_{s_2}}{\int_{s_2}}\right)k^2,$$

$$C^I_{11} = \frac{C'_c}{6}\left(\frac{A^{1z}_{2z}A_1\int'_{s_1}}{\int_{s_1}} + \frac{A^{2z}_{2z}A_2\int'_{s_2}}{\int_{s_2}}\right)(k^2-h^2) + \frac{C'_c}{6}\left(\frac{A^{1z}_{2z}B_1\int'_{s_1}}{\int_{s_1}} + \frac{A^{2z}_{2z}B_2\int'_{s_2}}{\int_{s_2}}\right)k^2,$$

with

$$s_1 = \tfrac{1}{6}(p_1 - 4)(h^2 + k^2),\ s_2 = \tfrac{1}{6}(p_2 - 4)(h^2 + k^2),$$

$$C_c = \frac{16\pi k_1(k_2 - k_1)}{15},\ C'_c = \frac{2(k_2 - k_1)}{3},$$

$$C_1 = 2k^2(h^2 + s_1) - (k^2 - h^2)s_1,\quad C_2 = 2k^2(h^2 + s_2) - (k^2 - h^2)s_2,$$

$$C_3 = 2(k^2 - h^2)s_1 - k^2(h^2 + s_1),\quad C_4 = 2(k^2 - h^2)s_2 - k^2(h^2 + s_2),$$

$$C_5 = -k^2(h^2 + s_1) - (k^2 - h^2)s_1,\ C_6 = -k^2(h^2 + s_2) - (k^2 - h^2)s_2,$$

$$A_1 = s_1(k^2 + s_1),\ A_2 = s_2(k^2 + s_2),$$

$$B_1 = (k^2 + s_1)(h^2 + s_1),\ B_2 = (k^2 + s_2)(h^2 + s_2),$$

$$\int_{s_1} = \int_\lambda^\infty \frac{\mathrm{d}t}{(a^2 + s_1 + t)^2\Delta(t)}(k_2 - k_1) + \frac{k_1}{(a^2 + s_1)abc},$$

$$\int_{s_2} = \frac{\mathrm{d}t}{(a^2 + s_2 + t)^2\Delta(t)}(k_2 - k_1) + \frac{k_1}{(a^2 + s_2)abc},$$

$$\int'_{s_1} = \int_{\lambda}^{\infty} \frac{dt}{(a^2 + s_1 + t)^2 \Delta(t)},$$

$$\int'_{s_2} = \int_{\lambda}^{\infty} \frac{dt}{(a^2 + s_2 + t)^2 \Delta(t)},$$

$$A_{2x}^{1x} = \frac{3s_2}{h^2k^2(s_2 - s_1)}, \; A_{2x}^{2x} = \frac{-3s_1}{h^2k^2(s_2 - s_1)},$$

$$A_{2y}^{1y} = \frac{-3(s_2 + h^2)}{h^2(k^2 - h^2)(s_2 - s_1)}, \; A_{2y}^{2y} = \frac{3(s_1 + h^2)}{h^2(k^2 - h^2)(s_2 - s_1)},$$

$$A_{2z}^{1z} = \frac{3(s_2 + k^2)}{k^2(k^2 - h^2)(s_2 - s_1)}, \; A_{2z}^{2z} = \frac{-3(s_1 + k^2)}{k^2(k^2 - h^2)(s_2 - s_1)}.$$

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